On the presentation of pointed Hopf algebras*

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March 31, 2010

Abstract

We give a presentation in terms of generators and relations of Hopf algebras generated by skew-primitive elements and abelian group of group-like elements with action given via characters. This class of pointed Hopf algebras has shown great importance in the classification theory and can be seen as generalized quantum groups. As a consequence we get an analog presentation of Nichols algebras of diagonal type.

KEY WORDS: Hopf algebra, Nichols algebra, quantum group

Introduction

Many famous examples of Hopf algebras generated by skew-primitive elements and abelian group of group-like elements with action given via characters are known: the universal enveloping algebras $U(\mathfrak{g})$ of a semi-simple complex Lie algebra $\mathfrak{g}$, their q-deformations $U_q(\mathfrak{g})$ the quantum groups of Drinfel'd and Jimbo, and the small quantum groups $u_q(\mathfrak{g})$ of Lusztig [14, 15], also called Frobenius-Lusztig kernels. These are all pointed, i.e., all its simple subcoalgebras are one-dimensional, or equivalently, the coradical equals the group algebra of the group of group-like elements.

Moreover, this class of pointed Hopf algebras is very important in the classification theory: It is conjectured that any finite-dimensional pointed Hopf algebra over the complex numbers with abelian coradical is of that type. Recently Andruskiewitsch and Schneider [5] have proven this with some restriction on the order of the group of group-like elements. Further it is true in other special cases: for cocommutative Hopf algebras we have the Cartier-Kostant-Milnor-Moore theorem of around 1963, if the Hopf algebra has rank one see [12], and if the dimension of the Hopf algebra is some power of a prime see [1, 2, 3, 6]. However, the setting in the general situation is more complicated and new phenomena appear; for concrete examples see [7, 8].

In Theorem 3.7 we give a structural description of Hopf algebras generated by skew-primitive elements and abelian group of group-like elements with action given via characters, in terms of generators and relations - we also get an analog statement for Nichols algerbas of diagonal type: At first we define a smash product of a free algebra and a group algebra, which is the prototype of these Hopf algebras. Then we show that any such Hopf algebra is a quotient of this prototype. The main point is the construction of the ideal, where we use the theory of Lyndon words. We then use a result by Kharchenko [11]. Also

*This work is part of the author's PhD thesis written under the supervision of Professor H.-J. Schneider.

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the generators of the ideal build up a Gröbner basis, see also [9]. The knowledge of this presentation is important, if one wants to determine the liftings of Nichols algebras of diagonal type [8], which is part of the lifting method by Andruskiewitsch and Schneider [1].

The paper is organized as follows: In Section 1 we recall the general calculus for q-commutators in an arbitrary algebra of [9]. Then in Section 2 we give a short account to the theory of Lyndon words, super letters and super words; super letters are iterated q-commutators and super words are products of super letters. We show that the set of all super words can be seen indeed as a set of words, i.e., as a free monoid. This is a consequence of Proposition 2.6. Finally, we formulate in Section 3 our main result, and give some applications and classical examples in Section 4; more complicate examples are found in [7, 8].

Throughout the paper let $\Bbbk$ denote a field of arbitrary characteristic $\operatorname{char}\Bbbk = p \geq 0$, unless stated otherwise.

1 q-commutator calculus

In this section let A denote an arbitrary algebra over $\Bbbk$. For all $a, b \in A$ and $q \in \Bbbk$ we define the *q-commutator*

$$[a, b]_q := ab - qba.$$

The q-commutator is bilinear. If $q = 1$ we get the classical commutator of an algebra. If A is graded and a, b are homogeneous elements, then there is a natural choice for the q. We are interested in the following special case:

Example 1.1. Let $\theta \geq 1$, $X = \{x_1, \ldots, x_\theta\}$, $\langle X\rangle$ the free monoid and $A = \Bbbk\langle X\rangle$ the free $\Bbbk$-algebra. For an abelian group Γ let $\widehat{\Gamma}$ be the character group, $g_1, \ldots, g_\theta \in \Gamma$ and $\chi_1, \ldots, \chi_\theta \in \widehat{\Gamma}$. If we define the two monoid maps

$$\deg_\Gamma : \langle X\rangle \to \Gamma,\ \deg_\Gamma(x_i) := g_i \quad \text{and} \quad \deg_{\widehat{\Gamma}} : \langle X\rangle \to \widehat{\Gamma},\ \deg_{\widehat{\Gamma}}(x_i) := \chi_i,$$

for all $1 \leq i \leq \theta$, then $\Bbbk\langle X\rangle$ is Γ- and $\widehat{\Gamma}$-graded. Let $a \in \Bbbk\langle X\rangle$ be Γ-homogeneous and $b \in \Bbbk\langle X\rangle$ be $\widehat{\Gamma}$-homogeneous. We set

$$g_a := \deg_\Gamma(a), \quad \chi_b := \deg_{\widehat{\Gamma}}(b), \quad \text{and} \quad q_{a,b} := \chi_b(g_a).$$

Further we define $\Bbbk$-linearly on $\Bbbk\langle X\rangle$ the q-commutator

$$[a, b] := [a, b]_{q_{a,b}}. \tag{1.1}$$

Note that $q_{a,b}$ is a bicharacter on the homogeneous elements and depends only on the values

$$q_{ij} := \chi_j(g_i) \text{ with } 1 \leq i, j \leq \theta.$$

For example $[x_1, x_2] = x_1x_2 - \chi_2(g_1)x_2x_1 = x_1x_2 - q_{12}x_2x_1$. Further if a, b are $\mathbb{Z}^\theta$-homogeneous they are both Γ- and $\widehat{\Gamma}$-homogeneous. In this case we can build iterated q-commutators, like $\big[x_1, [x_1, x_2]\big] = x_1[x_1, x_2] - \chi_1\chi_2(g_1)[x_1, x_2]x_1 = x_1[x_1, x_2] - q_{11}q_{12}[x_1, x_2]x_1$.

Later we will deal with algebras which still are $\widehat{\Gamma}$-graded, but not Γ-graded such that Eq. (1.1) is not well-defined. However, the q-commutator calculus, which we next want to develop, will be a major tool for our calculations such that we need the general definition with the q as an index.

Proposition 1.2. [9, Prop. 1.2] *For all $a, b, c, a_i, b_i \in A$, $q, q', q'', q_i \in \Bbbk$ with $1 \le i \le n$ we have:*

(1) q-derivation properties:

$$[a, bc]_{qq'} = [a, b]_q c + qb[a, c]_{q'}, \qquad [ab, c]_{qq'} = a[b, c]_{q'} + q'[a, c]_q b,$$

$$[a, b_1 \dots b_n]_{q_1 \dots q_n} = \sum_{i=1}^{n} q_1 \dots q_{i-1} b_1 \dots b_{i-1} [a, b_i]_{q_i} b_{i+1} \dots b_n,$$

$$[a_1 \dots a_n, b]_{q_1 \dots q_n} = \sum_{i=1}^{n} q_{i+1} \dots q_n a_1 \dots a_{i-1} [a_i, b]_{q_i} a_{i+1} \dots a_n.$$

(2) q-Jacobi identity:

$$\big[[a, b]_{q'}, c\big]_{q''q} = \big[a, [b, c]_q\big]_{q'q''} - q'b[a, c]_{q''} + q[a, c]_{q''} b.$$

Remark 1.3. If we are in the situation of Example 1.1 and assume that the elements are homogeneous, we can replace the arbitrary commutators by Eq. (1.1) and also replace the general q's above in the obvious way; e.g., in the first one of (1) set $q = q_{a,b}$, $q' = q_{a,c}$ and in (3), (4) $\zeta = q_{b,b}$ resp. $\zeta = q_{a,a}$.

2 Lyndon words and q-commutators

In this section we recall the theory of Lyndon words [13, 16] as far as we are concerned and then introduce the notion of super letters and super words [11]. We want to emphasize that the set of all super words can be seen indeed as a set of words (more exactly as a free monoid, see [9]), which is a consequence of Proposition 2.6. Moreover, we introduce a well-founded ordering of the super words.

2.1 Words and the lexicographical order

Let $\theta \ge 1$, $X = \{x_1, x_2, \dots, x_\theta\}$ be a finite totally ordered set by $x_1 < x_2 < \dots < x_\theta$, and $\langle X \rangle$ the free monoid; we think of X as an alphabet and of $\langle X \rangle$ as the words in that alphabet including the empty word 1. For a word $u = x_{i_1} \dots x_{i_n} \in \langle X \rangle$ we define $\ell(u) := n$ and call it the *length* of u.

The *lexicographical order* $\le$ on $\langle X \rangle$ is defined for $u, v \in \langle X \rangle$ by $u < v$ if and only if either v begins with u, i.e., $v = uv'$ for some $v' \in \langle X \rangle \backslash \{1\}$, or if there are $w, u', v' \in \langle X \rangle$, $x_i, x_j \in X$ such that $u = wx_iu'$, $v = wx_jv'$ and $i < j$. E.g., $x_1 < x_1x_2 < x_2$. This order $<$ is stable by left, but in general not stable by right multiplication: $x_1 < x_1x_2$ but $x_1x_3 > x_1x_2x_3$. Still we have:

Lemma 2.1. *Let $v, w \in \langle X \rangle$ with $v < w$. Then:*

(1) *$uv < uw$ for all $u \in \langle X \rangle$.*

(2) *If w does not begin with v, then $vu < wu'$ for all $u, u' \in \langle X \rangle$.*

2.2 Lyndon words and the Shirshov decomposition

A word $u \in \langle X \rangle$ is called a *Lyndon word* if $u \neq 1$ and u is smaller than any of its proper endings, i.e., for all $v, w \in \langle X \rangle \backslash \{1\}$ such that $u = vw$ we have $u < w$. We denote by

$$\mathcal{L} := \{u \in \langle X \rangle \mid u \text{ is a Lyndon word}\}$$

the set of all Lyndon words. For example $X \subset \mathcal{L}$, but $x_i^n \notin \mathcal{L}$ for all $1 \leq i \leq \theta$ and $n \geq 2$. Moreover, if $i < j$ then $x_i^n x_j^m \in \mathcal{L}$ for $n, m \geq 1$, e.g. x_1x_2, $x_1x_1x_2$, $x_1x_2x_2$, $x_1x_1x_2x_2$; also $x_i(x_ix_j)^n \in \mathcal{L}$ for any $n \in \mathbb{N}$, e.g. $x_1x_1x_2$, $x_1x_1x_2x_1x_2$.

For any $u \in \langle X \rangle \backslash X$ we call the decomposition $u = vw$ with $v, w \in \langle X \rangle \backslash \{1\}$ such that w is the minimal (with respect to the lexicographical order) ending the *Shirshov decomposition* of the word u. We will write in this case

$$\mathrm{Sh}(u) = (v|w).$$

E.g., $\mathrm{Sh}(x_1x_2) = (x_1|x_2)$, $\mathrm{Sh}(x_1x_1x_2x_1x_2) = (x_1x_1x_2|x_1x_2)$, $\mathrm{Sh}(x_1x_1x_2) \neq (x_1x_1|x_2)$.

If $u \in \mathcal{L} \backslash X$, this is equivalent to w is the longest proper ending of u such that $w \in \mathcal{L}$. Moreover we have another characterization of the Shirshov decomposition of Lyndon words:

Theorem 2.2. [13, Prop. 5.1.3, 5.1.4] *Let $u \in \langle X \rangle \backslash X$ and $u = vw$ with $v, w \in \langle X \rangle$. Then the following are equivalent:*

(1) *$u \in \mathcal{L}$ and $\mathrm{Sh}(u) = (v|w)$.*

(2) *$v, w \in \mathcal{L}$ with $v < u < w$ and either $v \in X$ or else if $\mathrm{Sh}(v) = (v_1|v_2)$ then $v_2 \geq w$.*

With this property we see that any Lyndon word is a product of two other Lyndon words of smaller length. Hence we get every Lyndon word by starting with X and concatenating inductively each pair of Lyndon words v, w with $v < w$.

Definition 2.3. We call a subset $L \subset \mathcal{L}$ *Shirshov closed* if $X \subset L$, and for all $u \in L$ with $\mathrm{Sh}(u) = (v|w)$ also $v, w \in L$.

For example $\mathcal{L}$ is Shirshov closed, and if $X = \{x_1, x_2\}$, then $\{x_1, x_1x_1x_2, x_2\}$ is not Shirshov closed, whereas $\{x_1, x_1x_2, x_1x_1x_2, x_2\}$ is. Later we will need the following:

Lemma 2.4. [11, Lem. 4] *Let $u, v \in \mathcal{L}$ and $u_1, u_2 \in \langle X \rangle \backslash \{1\}$ such that $u = u_1u_2$ and $u_2 < v$. Then we have*

$$uv < u_1v < v \text{ and } uv < u_2v < v.$$

2.3 Super letters and super words

Let the free algebra $\Bbbk\langle X \rangle$ be graded as in Example 1.1. For any $u \in \mathcal{L}$ we define recursively on $\ell(u)$ the map

$$[\,.\,] : \mathcal{L} \to \Bbbk\langle X \rangle, \quad u \mapsto [u]. \tag{2.1}$$

If $\ell(u) = 1$, then set $[x_i] := x_i$ for all $1 \leq i \leq \theta$. Else if $\ell(u) > 1$ and $\mathrm{Sh}(u) = (v|w)$ we define $[u] := \big[[v], [w]\big]$. This map is well-defined since inductively all $[u]$ are $\mathbb{Z}^\theta$-homogeneous such that we can build iterated q-commutators; see Example 1.1. The elements $[u] \in$

$\Bbbk\langle X\rangle$ with $u \in \mathcal{L}$ are called *super letters*. E.g. $[x_1x_1x_2x_1x_2] = \big[[x_1x_1x_2],[x_1x_2]\big] = \big[[x_1,[x_1,x_2]],[x_1,x_2]\big]$. If $L \subset \mathcal{L}$ is Shirshov closed then the subset of $\Bbbk\langle X\rangle$

$$[L] := \big\{[u] \,\big|\, u \in L\big\}$$

is a set of iterated q-commutators. Further $[\mathcal{L}] = \big\{[u] \,\big|\, u \in \mathcal{L}\big\}$ is the set of all super letters and the map $[\,.\,] : \mathcal{L} \to [\mathcal{L}]$ is a bijection, which follows from Lemma 2.5 below. Hence we can define an order $\leq$ of the super letters $[\mathcal{L}]$ by

$$[u] < [v] :\Leftrightarrow u < v,$$

thus $[\mathcal{L}]$ is a new alphabet containing the original alphabet X; so the name "letter" makes sense. Consequently, products of super letters are called *super words*. We denote

$$[\mathcal{L}]^{(\mathbb{N})} := \big\{[u_1]\dots[u_n] \,\big|\, n \in \mathbb{N},\, u_i \in \mathcal{L}\big\}$$

the subset of $\Bbbk\langle X\rangle$ of all super words. In order to define a lexicographical order on $[\mathcal{L}]^{(\mathbb{N})}$, we need to show that an arbitrary super word has a unique factorization in super letters. This is not shown in [11].

For any word $u = x_{i_1}x_{i_2}\dots x_{i_n} \in \langle X\rangle$ we define the *reversed word* $\overleftarrow{u} := x_{i_n}\dots x_{i_2}x_{i_1}$. Clearly, $\overleftarrow{\overleftarrow{u}} = u$ and $\overleftarrow{uv} = \overleftarrow{v}\,\overleftarrow{u}$. Further for any $a = \sum \alpha_i u_i \in \Bbbk\langle X\rangle$ we call the lexicographically smallest word of the u_i with $\alpha_i \neq 0$ the *leading word* of a and further define $\overleftarrow{a} := \sum \alpha_i \overleftarrow{u_i}$.

Lemma 2.5. *Let $u \in \mathcal{L}\backslash X$. Then there exist $n \in \mathbb{N}$, $u_i \in \langle X\rangle$, $\alpha_i \in \Bbbk$ for all $1 \leq i \leq n$ and $q \in \Bbbk^\times$ such that*

$$[u] = u + \sum_{i=0}^{n} \alpha_i u_i + q\overleftarrow{u} \quad \textit{and} \quad \overleftarrow{[u]} = \overleftarrow{u} + \sum_{i=0}^{n} \alpha_i \overleftarrow{u_i} + qu.$$

Moreover, u is the leading word of both $[u]$ and $\overleftarrow{[u]}$.

Proof. We proceed by induction on $\ell(u)$. If $\ell(u) = 2$, then $u = x_ix_j$ for some $1 \leq i < j \leq \theta$ and $[u] = [x_ix_j] = x_ix_j - q_{ij}x_jx_i = u - q_{ij}\overleftarrow{u}$. Let $\ell(u) > 2$, $\mathrm{Sh}(u) = (v|w)$ and $[u] = [v][w] - q_{vw}[w][v]$. By induction

$$[v] = v + \sum_i \beta_i v_i + q\overleftarrow{v} \quad \text{and} \quad \overleftarrow{[v]} = \overleftarrow{v} + \sum_i \beta_i \overleftarrow{v_i} + qv, \quad \text{resp.}$$
$$[w] = w + \sum_j \gamma_i w_i + q'\overleftarrow{w} \quad \text{and} \quad \overleftarrow{[w]} = \overleftarrow{w} + \sum_i \gamma_i \overleftarrow{w_i} + q'w$$

with $q, q' \neq 0$ and leading word v resp. w. Hence $[v][w]$ and $\overleftarrow{[v]}\overleftarrow{[w]}$ resp. $[w][v]$ and $\overleftarrow{[w]}\overleftarrow{[v]}$ have the leading words vw resp. wv. Since u is Lyndon we get $u = vw < wv$, thus the leading word of $[u]$ and $\overleftarrow{[u]}$ is u and further they are of the claimed form. $\square$

Proposition 2.6. *Let $u_1, \dots, u_n, v_1, \dots, v_m \in \mathcal{L}$. If $[u_1][u_2]\dots[u_n] = [v_1][v_2]\dots[v_m]$, then $m = n$ and $u_i = v_i$ for all $1 \leq i \leq n$.*

Proof. Induction on $\max\{m, n\}$, we may suppose $m \leq n$. If $n = 1$ then also $m = 1$, hence $[u_1] = [v_1]$ and both have the same leading word $u_1 = v_1$.

Let $n > 1$: By Lemma 2.5 $[u_1]\dots[u_n] = [v_1]\dots[v_m]$ has the leading word $u_1 \dots u_n = v_1 \dots v_m$ and

$$\overleftarrow{[u_n]}\dots\overleftarrow{[u_1]} = \overleftarrow{[u_1]\dots[u_n]} = \overleftarrow{[v_1]\dots[v_m]} = \overleftarrow{[v_m]}\dots\overleftarrow{[v_1]}$$

has the leading word $u_n \dots u_1 = v_m \dots v_1$.

If $\ell(u_1) \geq \ell(v_1)$, then $u_1 = v_1 u$ and $u_1 = u' v_1$ for some $u, u' \in \langle X \rangle$. If $u, u' \neq 1$, we get the contradiction $v_1 < v_1 u = u' v_1 < v_1$, since u_1 is Lyndon. Else if $\ell(u_1) < \ell(v_1)$, it is the same argument using that v_1 is Lyndon. Hence $u_1 = v_1$ and by induction the claim follows. $\square$

Now the lexicographical order on all super words $[\mathcal{L}]^{(\mathbb{N})}$, as defined above on regular words, is well-defined. We denote it also by $\leq$.

2.4 A well-founded ordering of super words

The *length* of a super word $U = [u_1][u_2]\dots[u_n] \in [L]^{(\mathbb{N})}$ is defined as $\ell(U) := \ell(u_1 u_2 \dots u_n)$.

Definition 2.7. For $U, V \in [\mathcal{L}]^{(\mathbb{N})}$ we define $U \prec V$ by

- $\ell(U) < \ell(V)$, or
- $\ell(U) = \ell(V)$ and $U > V$ lexicographically in $[\mathcal{L}]^{(\mathbb{N})}$.

This defines a total ordering of $[\mathcal{L}]^{(\mathbb{N})}$ with minimal element 1. As X is assumed to be finite, there are only finitely many super letters of a given length. Hence every nonempty subset of $[\mathcal{L}]^{(\mathbb{N})}$ has a minimal element, or equivalently, $\preceq$ fulfills the descending chain condition: $\preceq$ is *well-founded*, making way for inductive proofs on $\preceq$.

3 A class of pointed Hopf algebras

In this chapter we deal with a special class of pointed Hopf algebras. Let us recall the notions and results of [11, Sect. 3]: Let $\theta \geq 1$. A Hopf algebra A is called a *character Hopf algebra* if it is generated as an algebra by elements $a_1, \dots, a_\theta$ and an abelian group $G(A) = \Gamma$ of all group-like elements such that for all $1 \leq i \leq \theta$ there are $g_i \in \Gamma$ and $\chi_i \in \widehat{\Gamma}$ with

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i \qquad \text{and} \qquad g a_i = \chi_i(g) a_i g.$$

As mentioned in the introduction this covers a wide class of examples of Hopf algebras. The minimal number θ such that $a_1, \dots, a_\theta$ (also with renumbering) and Γ generate A is called the *rank* of A.

The aim of this section is to construct for any *character Hopf algebra* A a smash product $\Bbbk\langle X \rangle \# \Bbbk[\Gamma]$ together with an ideal I such that $A \cong (\Bbbk\langle X \rangle \# \Bbbk[\Gamma])/I$. Note that any character Hopf algebra is $\widehat{\Gamma}$-graded by $A = \oplus_{\chi \in \widehat{\Gamma}} A^\chi$ with $A^\chi := \{a \in A \mid ga = \chi(g) a g\}$, since A is genereated by $\widehat{\Gamma}$-homogeneous elements, and elements of different A^χ are linearly independent.

3.1 PBW basis in hard super letters

Let from now on A be a character Hopf algebra. The algebra map

$$\Bbbk\langle X\rangle \to A, \quad x_i \mapsto a_i$$

allows to identify elements of $\Bbbk\langle X\rangle$ with elements of A: By abuse of language we will write for the image of $a \in \Bbbk\langle X\rangle$ also a. Further let $\Bbbk\langle X\rangle$ be Γ-, $\widehat{\Gamma}$-graded and $q_{u,v}$ as in Example 1.1 with the g_i and χ_i above. Then a super letter $[u] \in A$ is called *hard* if it is not a linear combination of

- $U = [u_1]\dots[u_n] \in [\mathcal{L}]^{(\mathbb{N})}$ with $n \geq 1$, $\ell(U) = \ell(u)$, $u_i > u$ for all $1 \leq i \leq n$, and
- Vg with $V \in [\mathcal{L}]^{(\mathbb{N})}$, $\ell(V) < \ell(u)$ and $g \in \Gamma$.

Note that if $[u]$ is hard and $\mathrm{Sh}(u) = (v|w)$, then also $[v]$ and $[w]$ are hard; this follows from [11, Cor. 2]. We may assume that $a_1, \dots, a_\theta$ are hard, otherwise A would be generated by Γ and a proper subset of $a_1, \dots, a_\theta$. But this says that the set of all hard super letters is Shirshov closed.

For any hard $[u]$ we define $N'_u \in \{2, 3, \dots, \infty\}$ as the minimal $r \in \mathbb{N}$ such that $[u]^r$ is not a linear combination of

- $U = [u_1]\dots[u_n] \in [\mathcal{L}]^{(\mathbb{N})}$ with $n \geq 1$, $\ell(U) = r\ell(u)$, $u_i > u$ for all $1 \leq i \leq n$, and
- Vg with $V \in [\mathcal{L}]^{(\mathbb{N})}$, $\ell(V) < r\ell(u)$ and $g \in \Gamma$.

Theorem 3.1. [11, Thm. 2, Lem. 13] *Let A be a character Hopf algebra. Then the set of all*

$$[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t}g$$

with $t \in \mathbb{N}$, $[u_i]$ is hard, $u_1 > \dots > u_t$, $0 < r_i < N'_{u_i}$, $g \in \Gamma$, forms a $\Bbbk$-basis of A.

Further, for every hard super letter $[u]$ with $N'_u < \infty$ we have $\mathrm{ord} q_{u,u} = N'_u$ if $\mathrm{char}\,\Bbbk = 0$ resp. $p^k \mathrm{ord} q_{u,u} = N'_u$ for some $k \geq 0$ if $\mathrm{char}\,\Bbbk = p > 0$.

We now generally construct a smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ with an ideal I.

3.2 Prototype: The smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$

Let $\Bbbk\langle X\rangle$ be Γ- and $\widehat{\Gamma}$-graded as in Example 1.1, and $\Bbbk[\Gamma]$ be endowed with the usual bialgebra structure $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in \Gamma$. Then we define

$$g \cdot x_i := \chi_i(g)x_i, \quad \text{for all } 1 \leq i \leq \theta.$$

In this case, $\Bbbk\langle X\rangle$ is a $\Bbbk[\Gamma]$-module algebra and we calculate $gx_i = \chi_i(g)x_ig$, $gh = hg = \varepsilon(g)hg$ in $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$. Thus $x_i \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_i}$ and $\Bbbk[\Gamma] \subset (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\varepsilon}$ and in this way $\Bbbk\langle X\rangle\#\Bbbk[\Gamma] = \oplus_{\chi\in\widehat{\Gamma}}(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi}$. This $\widehat{\Gamma}$-grading extends the $\widehat{\Gamma}$-grading of $\Bbbk\langle X\rangle$ in Example 1.1 to $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$.

Further $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ is a Hopf algebra with structure determined by

$$\Delta(x_i) := x_i \otimes 1 + g_i \otimes x_i \qquad \text{and} \qquad \Delta(g) := g \otimes g,$$

for all $1 \leq i \leq \theta$ and $g \in \Gamma$. Thus our prototype is indeed a character Hopf algebra. Other character Hopf algebras arise from certain quotients of this prototype, as an example consider the following:

3.3 Motivation: Quantum groups $U_q(\mathfrak{sl}_2)$ and $u_q(\mathfrak{sl}_2)$

Let $q \in \Bbbk^\times \backslash \{\pm 1\}$, then we define like in [10, Sect. VI,VII]

$$U_q(\mathfrak{sl}_2) := \Bbbk\Big\langle E, F, K, K^{-1} \;\Big|\; KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\Big\rangle,$$
$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$

which is a character Hopf algebra, too. Let us rewrite this presentation to our conventions:

Example 3.2. *Quantum group.* Set $\Gamma := \langle g, g^{-1} \mid gg^{-1} = g^{-1}g = 1\rangle \cong \mathbb{Z}$, $g_1 := g_2 := g$, and $\chi_1(g) := q^{-2}$, $\chi_2(g) := q^2$. Then

$$U_q(\mathfrak{sl}_2) \cong \big(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma]\big) / \big([x_1x_2] - (1 - g^2)\big).$$

Proof. The isomorphism of Hopf algebras is the map from $U_q(\mathfrak{sl}_2)$ to the right-hand side which sends $E \mapsto \frac{q^2}{q-q^{-1}} x_1 g^{-1}$, $F \mapsto x_2$, $K \mapsto g^{-1}$, $K^{-1} \mapsto g$. □

A finite-dimensional version is the following example:

Example 3.3. *Frobenius-Lusztig kernel.* Let $q \in \Bbbk^\times$ with odd $\operatorname{ord} q = N > 2$, $\Gamma := \langle g \mid g^N = 1\rangle \cong \mathbb{Z}/(N)$, $g_1 := g_2 := g$, and $\chi_1(g) := q^{-2}$, $\chi_2(g) := q^2$. Then

$$u_q(\mathfrak{sl}_2) \cong \big(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma]\big) / \big(\ [x_1x_2] - (1 - g^2),\ x_1^N,\ x_2^N\ \big).$$

Note that in the above examples there are relations involving super letters and powers of super letters. Next we construct the ideals in the general setting:

3.4 Ideals associated to Shirshov closed sets

In this subsection we fix a Shirshov closed $L \subset \mathcal{L}$. We want to introduce the following notation for an $a \in \Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ and $W \in [\mathcal{L}]^{(\mathbb{N})}$: We will write $a \prec_L W$ (resp. $a \preceq_L W$), if a is a linear combination of

- $U \in [L]^{(\mathbb{N})}$ with $\ell(U) = \ell(W)$, $U > W$ (resp. $U \geq W$), and
- Vg with $V \in [L]^{(\mathbb{N})}$, $g \in \Gamma$, $\ell(V) < \ell(W)$.

Furthermore, we set for each $u \in L$ either $N_u := \infty$ or $N_u := \operatorname{ord} q_{u,u}$ (resp. $N_u := p^k \operatorname{ord} q_{u,u}$ with $k \geq 0$ if $\operatorname{char} \Bbbk = p > 0$) and we want to distinguish the following two sets of words depending on L:

$$C(L) := \big\{w \in \langle X\rangle \backslash L \mid \exists u, v \in L : w = uv,\ u < v,\ \text{and } \operatorname{Sh}(w) = (u|v)\big\},$$
$$D(L) := \big\{u \in L \mid N_u < \infty\big\}.$$

Note that $C(L) \subset \mathcal{L}$ and $D(L) \subset L \subset \mathcal{L}$ are sets of Lyndon words.

Moreover, let $c_w \in (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])^{\chi_w}$ for all $w \in C(L)$ such that $c_w \prec_L [w]$; and let $d_u \in (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])^{\chi_u^{N_u}}$ for all $u \in D(L)$ such that $d_u \prec_L [u]^{N_u}$. Now we define the $\widehat{\Gamma}$-homogeneous ideal:

Definition 3.4. In the above setting let $I_{L,N,c,d}$ be the ideal of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ generated by the following elements:

$$[w] - c_w \qquad \text{for all } w \in C(L), \tag{3.1}$$
$$[u]^{N_u} - d_u \qquad \text{for all } u \in D(L). \tag{3.2}$$

For shortness we will write just I for $I_{L,N,c,d}$.

Example 3.5. Let $X = \{x_1, x_2\}$. If $L = X$ then $C(L) = \{x_1x_2\}$ and we have Examples 3.2 and 3.3. It is $N_1 = N_2 = \infty$ in Example 3.2 and $N_1 = N_2 = N$ in 3.3. Other examples are treated in Sections 4.1 and 4.2. Furthermore we want to mention the list of more complicated examples of [7, 8].

In the next Lemma we want to define $c_{(u|v)} \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ for all $u, v \in L$ with $u < v$, such that $\big[[u], [v]\big] = c_{(u|v)}$ modulo I. This shows that the relations $\big[[u], [v]\big] = c_{(u|v)}$ with with $\mathrm{Sh}(uv) \neq (u|v)$ or $uv \in L$ are redundant modulo I.

Lemma 3.6. *Let $I' \subset \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ be the ideal generated by the elements Eq.* (3.1)*. Then there are $c_{(u|v)} \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_{uv}}$ for all $u, v \in L$ with $u < v$ such that*

(1) $\big[[u], [v]\big] - c_{(u|v)} \in I'$,

(2) $c_{(u|v)} \preceq_L [uv]$.

The residue classes of $[u_1]^{r_1}[u_2]^{r_2}\ldots[u_t]^{r_t}g$ with $t \in \mathbb{N}$, $u_i \in L$, $u_1 > \ldots > u_t$, $0 < r_i < N_{u_i}$, $g \in \Gamma$, $\Bbbk$-generate $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I$.

Proof. For all $u, v \in L$ with $u < v$ and $\mathrm{Sh}(uv) = (u|v)$ we set

$$c_{(u|v)} := \begin{cases} [uv], & \text{if } uv \in L, \\ c_{uv}, & \text{if } uv \notin L. \end{cases}$$

We then proceed by induction on $\ell(u)$: If $u \in X$ then $\mathrm{Sh}(uv) = (u|v)$ by Theorem 2.2 and by definition the claim is fulfilled. So let $\ell(u) > 1$. Again if $\mathrm{Sh}(uv) = (u|v)$ then we argue as in the induction basis. Conversely, let $\mathrm{Sh}(uv) \neq (u|v)$, and further $\mathrm{Sh}(u) = (u_1|u_2)$; then $u_2 < v$ by Theorem 2.2 and by Lemma 2.4

$$u_1 < u_1u_2 = u < uv < u_2v, \text{ and } uv < u_1v. \tag{3.3}$$

By induction hypothesis there is a $c_{(u_2|v)} = \sum \alpha U + \sum \beta V g$ (we omit the indices to avoid double indices) of $\widehat{\Gamma}$-degree χ_{u_2v} with $U = [l_1]\ldots[l_n] \in [L]^{(\mathbb{N})}$, $\ell(U) = \ell(u_2v)$, $l_1 \geq u_2v$, $V \in [L]^{(\mathbb{N})}$, $\ell(V) < \ell(u_2v)$, $g \in \Gamma$ and $\big[[u_2], [v]\big] - c_{(u_2|v)} \in I'$. Then

$$\big[[u_1], c_{(u_2|v)}\big] = \sum \alpha\big[[u_1], U\big] + \sum \beta\big[[u_1], Vg\big].$$

Since U is χ_{u_2v}-homogeneous we can use the q-derivation property of Proposition 1.2 for the term

$$\big[[u_1], U\big] = \sum_{i=1}^{n} q_{u_1,l_1\ldots l_{i-1}}[l_1]\ldots[l_{i-1}]\big[[u_1], [l_i]\big][l_{i+1}]\ldots[l_n].$$

By assumption $u_2v \leq l_1$, hence we deduce $uv < l_1$ and $u_1 < l_1$ from Eq. (3.3); because of the latter inequality, by the induction hypothesis there is a $\chi_{u_1l_1}$-homogeneous $c_{(u_1|l_1)} =$

$\sum \alpha' U' + \sum \beta' V' g'$ with $U' \in [L]^{(\mathbb{N})}$, $\ell(U') = \ell(u_1 l_1)$, $U' \geq [u_1 l_1]$, $V' \in [L]^{(\mathbb{N})}$, $\ell(V') < \ell(u_1 l_1)$, $g' \in \Gamma$ and $\big[[u_1],[l_1]\big] - c_{(u_1|l_1)} \in I'$. Since $u_2 v \leq l_1$ we have $[uv] = [u_1 u_2 v] \leq [u_1 l_1] \leq U'$. We now define $\partial_{u_1}(c_{(u_2|v)})$ $\Bbbk$-linearly by

$$\partial_{u_1}(U) := c_{(u_1|l_1)}[l_2]\dots[l_n] + \sum_{i=2}^{n} q_{u_1,l_1\dots l_{i-1}}[l_1]\dots[l_{i-1}]\big[[u_1],[l_i]\big][l_{i+1}]\dots[l_n],$$

$$\partial_{u_1}(Vg) := \big[[u_1],V\big]_{q_{u_1,u_2v}\chi_{u_1}(g)} g.$$

Then $\partial_{u_1}(c_{(u_2|v)}) \preceq_L [uv]$ with $\widehat{\Gamma}$-degree χ_{uv}. Moreover $\big[[u_1],\big[[u_2],[v]\big]\big] - \partial_{u_1}(c_{(u_2|v)}) \in I'$, since $\big[[u_1],U\big] - \partial_{u_1}(U) \in I'$ and $\partial_{u_1}(Vg) = \big[[u_1],Vg\big]_{q_{u_1,u_2v}}$.

Finally, because of $u_1 < u < v$ there is again by induction assumption a $c_{(u_1|v)} \preceq_L [u_1 v]$, which is $\chi_{u_1 v}$-homogeneous and $c_{(u_1|v)} - \big[[u_1][v]\big] \in I'$ (moreover, $u_1 v > uv$ by Eq. (3.3)). We then define for $\mathrm{Sh}(uv) \neq (u|v)$

$$c_{(u|v)} := \partial_{u_1}(c_{(u_2|v)}) + q_{u_2,v} c_{(u_1|v)}[u_2] - q_{u_1,u_2}[u_2] c_{(u_1|v)}. \tag{3.4}$$

We have $u_2 > u$ since u is Lyndon and u cannot begin with u_2, hence $u_2 > uv$ by Lemma 2.1. Thus $c_{(u|v)} \prec_L [uv]$. Also $\deg_{\widehat{\Gamma}}(c_{(u|v)}) = \chi_{uv}$ and by the q-Jacobi identity of Proposition 1.2 we have $\big[[u],[v]\big] - c_{(u|v)} \in I'$.

For the last assertion it suffices to show that the residue classes of $[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t} g$ $\Bbbk$-generate the residue classes of $\Bbbk\langle X\rangle$ in $(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I'$: this can be done as in the proof of [11, Lem. 10] by induction on $\preceq$ using (1),(2). $\square$

3.5 Structure of pointed Hopf algebras and Nichols algebras

Theorem 3.7. *If A is a character Hopf algebra, then there is an ideal $I \subset \Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ as in Definition 3.4 such that*

$$A \cong (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I.$$

Proof. Let $[L]$ be the set of hard super letters in A; then $L \subset \mathcal{L}$ is Shirshov closed as mentioned above. By Theorem 3.1 the elements $[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t} g$ with $t \in \mathbb{N}$, $u_i \in L$, $u_1 > \dots > u_t$, $0 < r_i < N'_{u_i}$, $g \in \Gamma$, form a $\Bbbk$-basis. We consider the $\Bbbk$-linear map

$$\phi : A \to \Bbbk\langle X\rangle \# \Bbbk[\Gamma], \quad [u_1]^{r_1}\dots[u_t]^{r_t} g \mapsto [u_1]^{r_1}\dots[u_t]^{r_t} g,$$

and define $c_w := \phi\big([w]\big)$ for all $w \in C(L)$, $d_u := \phi\big([u]^{N_u}\big)$ for all $u \in D(L)$, where $N_u := N'_u$. Note that these elements are as stated in Lemma 3.6 since $[w]$ is not hard. Then we build the ideal $I \subset \Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ like in Definition 3.4 and there is the surjective Hopf algebra map

$$(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I \to A, \quad x_i \mapsto a_i, \; g \mapsto g.$$

By Lemma 3.6 the residue classes of $[u_1]^{r_1}\dots[u_t]^{r_t} g$ $\Bbbk$-generate $(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I$; they are linearly independent because so are their images. Hence the map is an isomorphism. $\square$

One immediately gets the following result for Nichols algebras of diagonal type (for the definition of Nichols algebras we refer to [4] or [8]):

Corollary 3.8. *Let $\mathfrak{B}(V)$ be a Nichols algebra of diagonal type of a vector space V with basis X. Then there is a homogeneous ideal $I \subset \Bbbk\langle X\rangle$ as in Definition 3.4 such that*

$$\mathfrak{B}(V) \cong \Bbbk\langle X\rangle/I.$$

4 Application and examples

We want to investigate the situation in more detail for character Hopf algebras of rank one and two for some fixed Shirshov closed subsets $L \subset \mathcal{L}$. Especially we want to treat liftings of Nichols algebras. Therefore we define the following scalars which will guarantee a $\widehat{\Gamma}$-graduation:

Definition 4.1. Let $L \subset \mathcal{L}$. Then we define coefficients $\mu_u \in \Bbbk$ for all $u \in D(L)$, and $\lambda_w \in \Bbbk$ for all $w \in C(L)$ by

$$\mu_u = 0, \text{ if } g_u^{N_u} = 1 \text{ or } \chi_u^{N_u} \neq \varepsilon, \qquad \lambda_w = 0, \text{ if } g_w = 1 \text{ or } \chi_w \neq \varepsilon,$$

and otherwise they can be chosen arbitrarily.

4.1 Pointed Hopf algebras of rank one

Proposition 4.2. *Let* $\operatorname{char}\Bbbk = p \geq 0$ *and* A *be a character Hopf algebra of rank one. Then either* $A \cong \Bbbk[x_1]\#\Bbbk[\Gamma]$, *or*

$$A \cong (\Bbbk[x_1]\#\Bbbk[\Gamma])/\big(x_1^N - d_1\big)$$

for $N = \mathrm{ord} q_{11} < \infty$ *if* $\operatorname{char}\Bbbk = 0$ *resp.* $N = p^k \mathrm{ord} q_{11} < \infty$ *with* $k \geq 0$ *if* $\operatorname{char}\Bbbk = p > 0$ *and* $d_1 \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_1^N}$ *with* $d_1 \prec_{\{x_1\}} x_1^N$.

Proof. We have $\mathcal{L} = X = \{x_1\}$. Hence any Shirshov closed L is equal to $\{x_1\}$ and thus $C(L)$ is the empty set. By Theorem 3.7 we get the claim. □

Corollary 4.3. *Let* $\operatorname{char}\Bbbk = p \geq 0$ *and* $\mathfrak{B}(V)$ *be a Nichols algebra of diagonal type with* $V = \Bbbk x_1$ *a one-dimensional vector space. Then either* $\mathfrak{B}(V) \cong \Bbbk[x_1]$, *or*

$$\mathfrak{B}(V) \cong \Bbbk[x_1]/\big(x_1^N\big)$$

for $N = \mathrm{ord} q_{11} < \infty$ *if* $\operatorname{char}\Bbbk = 0$ *resp.* $N = p^k \mathrm{ord} q_{11} < \infty$ *with* $k \geq 0$ *if* $\operatorname{char}\Bbbk = p > 0$.

Proof. By Theorem 3.7 the ideal is of the form $\big(x_1^N - d_1\big)$ with $d_1 \in \Bbbk\langle X\rangle^{\chi_1^N}$ and $d_1 \prec_{\{x_1\}} x_1^N$. Because x_1^N is a primitive element we get $d_1 = 0$, by definition of a Nichols algebra. □

If $\operatorname{char}\Bbbk = p$ not every finite-dimensional pointed Hopf algebra of rank one is a character Hopf algebra, since the group action is not necessarily via characters [17]. But we have:

Proposition 4.4. [12, Thm. 1] *If* $\operatorname{char}\Bbbk = 0$, *then any finite-dimensional pointed Hopf algebra of rank one is a character Hopf algebra; moreover it is isomorphic to*

$$(\Bbbk[x_1]\#\Bbbk[\Gamma])/\big(x_1^N - \mu_1(1 - g_1^N)\big)$$

with $N = \mathrm{ord} q_{11} < \infty$ *and* $\mu_1 \in \Bbbk$ *as in Definition 4.1.*

These are the liftings of the Nichols algebra of Cartan type A_1 of Corollary 4.3. As concrete realizations for $\Gamma = \mathbb{Z}/(N), \mathbb{Z}/(N^2)$ we name the following classic examples:

Examples 4.5. Let $\operatorname{char}\Bbbk = 0$.

1. *Taft Hopf algebra.* Let $\Gamma := \langle g_1 \mid g_1^N = 1\rangle \cong \mathbb{Z}/(N)$ and $\chi_1(g_1) := q \in \Bbbk^\times$ with $\mathrm{ord} q = N \geq 2$.
$$T(q) \;\cong\; \big(\Bbbk[x_1]\#\Bbbk[\Gamma]\big)/(x_1^N).$$
2. *Radford Hopf algebra.* Let $\Gamma := \langle g_1 \mid g_1^{N^2} = 1\rangle \cong \mathbb{Z}/(N^2)$ and $\chi_1(g_1) := q \in \Bbbk^\times$ with $\mathrm{ord} q = N \geq 2$.
$$r(q) \cong (\Bbbk[x_1]\#\Bbbk[\Gamma])/(x_1^N - (1 - g_1^N)).$$

4.2 Pointed Hopf algebras of rank two

If $X = \{x_1, x_2\}$, then the situation is much more complicated such that we will treat here only to the well-known case $L = X$.

More complicated and new examples of pointed Hopf algebras for $L = \{x_1, x_1x_2, x_2\}$, $\{x_1, x_1x_1x_2, x_1x_2, x_2\}$, etc., are found in [7, 8] as liftings of Nichols algebras.

Proposition 4.6. *Let* $\operatorname{char}\Bbbk = p \geq 0$. *Any character Hopf algebra* A *of rank 2 with hard super letters* $[L] = \{x_1, x_2\}$ *is isomorphic to*

$$(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma])/(\ [x_1x_2] - c_{12},\quad \begin{array}{l} x_1^{N_1} - d_1, \\ x_2^{N_2} - d_2\ ), \end{array}$$

with c_{12}, d_1, d_2 *as in Definition 3.4.*

Proof. Since $C(L) = \{x_1x_2\}$, the claim follows by Theorem 3.7. □

As an example let us name the Nichols algebra of Cartan type $A_1 \times A_1$ (quantum plane) and its liftings, with the following concrete realizations:

Examples 4.7. Let $\operatorname{char}\Bbbk = 0$.

1. *Nichols algebra of Cartan type* $A_1 \times A_1$. Let $q_{12}q_{21} = 1$, and $N_i = \operatorname{ord} q_{ii} \geq 2$, $i = 1, 2$.

$$\mathfrak{B}(V) \cong \Bbbk\langle x_1, x_2\rangle / \big(\ [x_1x_2],\ x_1^{N_1},\ x_2^{N_2}\ \big).$$

2. *Liftings of Cartan type* $A_1 \times A_1$. Let $q_{12}q_{21} = 1$, and $N_i = \operatorname{ord} q_{ii} \geq 2$, $i = 1, 2$.

$$\big(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma]\big)/\big(\quad [x_1x_2] - \lambda_{12}(1 - g_1g_2), \qquad \begin{array}{l} x_1^{N_1} - \mu_1(1 - g_1^{N_1}), \\ x_2^{N_2} - \mu_2(1 - g_2^{N_2}) \quad \big). \end{array}$$

3. *Book Hopf algebra.* Let $q \in \Bbbk^\times$ with $\operatorname{ord} q = N \geq 2$, $\Gamma := \langle g \mid g^N = 1\rangle \cong \mathbb{Z}/(N)$, $g_1 := g_2 := g$, and $\chi_1(g) := q^{-1}$, $\chi_2(g) := q$.

$$h(1, q) \cong \big(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma]\big)/\big(\ [x_1x_2],\ x_1^N,\ x_2^N\ \big).$$

4. *Quantum groups.* $U_q(\mathfrak{sl}_2)$ and $u_q(\mathfrak{sl}_2)$ of Section 3.3.

References

[1] N. Andruskiewitsch and H.-J. Schneider. Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. *J. Algebra*, 209:658–691, 1998.

[2] N. Andruskiewitsch and H.-J. Schneider. Finite quantum groups and Cartan matrices. *Adv. in Math.*, 154:1–45, 2000.

[3] N. Andruskiewitsch and H.-J. Schneider. Lifting of Nichols algebras of type A_2 and pointed Hopf algebras of order p^4. In S. Caenepeel and F. van Oystaeyen, editors, *Hopf algebras and quantum groups: Proceedings of the Brussels Conference*, volume 209 of *Lecture Notes in Pure and Appl. Math.*, pages 1–14. Marcel Dekker, 2000.

[4] N. Andruskiewitsch and H.-J. Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43, pages 1–68. MSRI Publications, Cambridge Univ. Press, 2002.

[5] N. Andruskiewitsch and H.-J. Schneider. On the classification of finite-dimensional pointed Hopf algebras. 2007. to appear in *Ann. Math.*, `arXiv math.QA/0502157`.

[6] M. Graña. On pointed Hopf algebras of dimension p^5. *Glasgow Math. J.*, 42:405–419, 2000.

[7] M. Helbig. *Lifting of Nichols algebras*. Südwestdeutscher Verlag für Hochschulschriften, 2009. available at `http://edoc.ub.uni-muenchen.de/10378/`.

[8] M. Helbig. On the lifting of Nichols algebras. preprint, available at `http://arxiv.org/`, 2010.

[9] M. Helbig. A PBW basis criterion for pointed Hopf algebras. preprint, available at `http://arxiv.org/`, 2010.

[10] C. Kassel. *Quantum groups*. Number 155 in Graduate Texts in Mathematics. Springer, 1995.

[11] V. Kharchenko. A quantum analog of the Poincaré-Birkhoff-Witt theorem. *Algebra and Logic*, 38:259–276, 1999.

[12] L. Krop and D. Radford. Finite-dimensional Hopf algebras of rank one in characteristic zero. *Journal of Algebra*, 302(1):214–230, 2006.

[13] M. Lothaire. *Combinatorics on Words*, volume 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983.

[14] G. Lusztig. Finite dimensional Hopf algebras arising from quantized universal enveloping algebras. *J. of Amer. Math. Soc.*, 3:257–296, 1990.

[15] G. Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35:89–114, 1990.

[16] C. Reutenauer. *Free Lie Algebras*, volume 7 of *London Mathematical Society Monographs, New Series*. Clarendon Press, London, 1993.

[17] S. Scherotzke. Classification of pointed rank one Hopf algebras. *Journal of Algebra*, 319(7):2889–2912, 2008.